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Orbital magnetism of a two-dimensional noncommutative confined system

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Abstract
We study a system of spinless electrons moving in a two-dimensional noncommutative space subject to a perpendicular magnetic field $\vec{B}$ and confined by a harmonic potential type $\frac{1}{2}mw_0r^2$. We look for the orbital magnetism of the electrons in different regimes of temperature T, magnetic field $\vec{B}$ and noncommutative parameter θ. We prove that the degeneracy of Landau levels can be lifted by the θ-term appearing in the electron energy spectrum at weak magnetic field. Using the Berezin–Lieb inequalities for thermodynamical potential, it is shown that in the high-temperature limit, the system exibits a magnetic θ-dependent behaviour, which is missing in the commutative case. Moreover, a correction to susceptibility at low T is observed. Using the Fermi–Dirac trace formulas, a generalization of the thermodynamical potential, the average number of electrons and the magnetization is obtained. There is a critical point where the thermodynamical potential becomes infinite in both of the two methods above. So at this point we deal with the partition function by adopting another approach. The standard results in the commutative case for this model can be recovered by switching off the θ-parameter.

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Dedicated to Professor Erdal Inönü on his 75 birthday

1. Introduction

It seems that the noncommutativity appeared in physics since Palev [1] investigated the noncanonical quantization of two particles interacting via a harmonic potential *à la* Wigner

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(see also [2–5]). One of the outcomes of his approach is that the position of any one of the particles cannot be localized in the space since the coordinates of particles do not commute

$$[\hat{r}_i, \hat{r}_j] \neq 0 \qquad i \neq j.$$

In field theories, the noncommutativity is introduced by replacing the standard product by the star product. For a manifold parametrized by the coordinates x^i, the noncommutative relation can be written as [6]

$$[x^i, x^j] = \mathrm{i}\theta^{ij} \tag{1}$$

where $\theta^{ij} = \epsilon^{ij}\theta$ is the noncommutative parameter and is of dimension of (length)^2, $\epsilon^{12} = -\epsilon^{21} = 1$. Basically, we are forced in this case to replace $fg(x) = f(x)g(x)$ by the relation

$$f(x) * g(x) = \exp\left[\frac{\mathrm{i}}{2}\theta^{ij}\partial_{x^i}\partial_{y^j}\right] f(x)g(y)|_{x=y} \tag{2}$$

where f and g are two arbitrary functions, supposed to be infinitely differentiable. The last equation defines the so-called Moyal bracket of functions

$$\{f, g\}_{MB} = f * g - g * f \tag{3}$$

which has been applied to solve some physical problems (for example, see [7]).

Recently, some applications of these mathematical tools were used to solve some physical problems. For instance, in quantum Hall effect a relation between θ and the quantized Hall conductivity has been established [8] and a study of the multi-skyrmions near the filling factor $\nu = 1$ has been done [9]. Furthermore, in hydrogen atom spectrum the energy levels have been analysed in the framework of noncommutativity [10]. Subsequently, with Dayi [11], we have considered the behaviour of electrons in an external uniform magnetic field $\vec{B}$, where the space coordinates perpendicular to $\vec{B}$ are taken as noncommuting. Calculating the susceptibility, we have found that the usual Landau diamagnetism is modified. We have also computed the susceptibility according to nonextensive statistics. We have found that these two methods agree under certain conditions. Basically, this paper [11] offers some possibilities to give a noncommutative description for any system showing an anomaly in the Boltzmann–Gibbs theory related to statistical physics.

On the other hand, orbital magnetism, which is possible only in quantum mechanics, has stimulated some work in this period (see [12] and references therein). With Gazeau *et al* [13], we have studied the possible occurrence of orbital magnetism for two-dimensional electrons confined by a harmonic potential in various regimes of temperature and magnetic field. Standard coherent state families are used for calculating symbols of various involved observables like the thermodynamical potential, magnetic moment or the spatial distribution of the current. Their expressions are given in a closed form and the resulting Berezin–Lieb inequalities provide a straightforward way to study magnetism in various limit regimes. In particular, we have predicted a paramagnetic behaviour in the thermodynamical limit as well as in the quasiclassical limit under a weak field. Finally, we have obtained an exact expression for the magnetic moment which yields a full description of the phase diagram of the magnetization.

Our main goal in this paper is to study the orbital magnetism of the model used in [13] in noncommutative space. Our idea is to consider a system of electrons moving on a noncommutative space and subject to a perpendicular magnetic field and harmonic confining potential. We show the differences of the commutative and noncommutative cases. In particular, employing the Berezin–Lieb inequalities we find that there is no degeneracy when the magnetic field is weak and point out a correction to susceptibility at low and high temperature T. Furthermore, using the Fermi–Dirac trace formulas, a general expression is derived for the thermodynamical potential, the average number of electrons and the magnetization. A critical point is found, such that at $\frac{eB\theta}{c} = -2$, the thermodynamical potential becomes infinite in both the methods mentioned above. However, by using another approach, we obtain the thermodynamical potential, which is found to be equivalent to that of 2d electrons in a uniform magnetic field. As a consequence we find infinite susceptibility for zero magnetic field.

The outline of the paper is as follows. In section 2, we give the noncommutative version of a Hamiltonian describing 2d electrons in the presence of a perpendicular magnetic field and confining potential. Using two different methods, we investigate the energy spectrum and the corresponding eigenfunctions in section 3. We study the degeneracy of Landau levels in section 4, where we also start with the realization of some algebras and investigate the magnetic field limits. In section 5, we derive the thermodynamical potential and the related physical quantities by using two methods: the first makes use of the Berezin–Lieb inequalities and the second one employs the Fermi–Dirac trace formulas. At critical point, we use another approach to obtain the thermodynamical potential and related quantities. The final section is devoted to conclusions and perspectives.

2. Electron in noncommutative space

Let us consider a system of spinless electrons (m, e) living on the (x, y)-space in a magnetic field $\vec{B}$. We recall that the eigenstates and eigenvalues were investigated for the first time by Landau [14]. When a harmonic confining potential is introduced and the Coulomb interactions are neglected, this system is described by the Fock–Darwin Hamiltonian [15–17]

$$H = \frac{1}{2m}\left(\vec{P} + \frac{e}{c}\vec{A}\right)^2 + \frac{1}{2}mw_0^2 r^2 \tag{4}$$

where $\vec{P}$ is the canonical momentum and $\vec{A}$ is the vector potential. We will study this Hamiltonian by making use of the commutation relations

$$[x^i, p^j] = \mathrm{i}\hbar\delta^{ij} \qquad [p^i, p^j] = 0 \tag{5}$$

as well as equation (1), and by choosing the symmetric gauge

$$\vec{A} = \left(-\frac{B}{2}y, \frac{B}{2}x\right). \tag{6}$$

According to this recipe, the above Hamiltonian acts on an arbitrary function $\Psi(\vec{r}, t)$ as

$$\begin{aligned} H * \Psi(\vec{r}, t) &= \frac{1}{2m}\left[\left(p_x - \frac{eB}{2c}y\right)^2 + \left(p_y + \frac{eB}{2c}x\right)^2 + m^2 w_0^2(x^2 + y^2)\right] * \Psi(\vec{r}, t) \\ &\equiv H_\theta \Psi(\vec{r}, t). \end{aligned} \tag{7}$$

Therefore, the noncommutative version of equation (4) can be written as follows:

$$H_\theta = \frac{1}{2m}\left(\left(\tilde{p}_x - \frac{eB}{2c}y\right)^2 + \left(\tilde{p}_y + \frac{eB}{2c}x\right)^2\right) + \frac{1}{2}mw_0^2\left(\left(\frac{1}{2}\theta p_x + y\right)^2 + \left(\frac{1}{2}\theta p_y - x\right)^2\right). \tag{8}$$

Here $\tilde{p}_\mu$ is a linear function of the noncommutative parameter, such that

$$\tilde{p}_\mu = \left(1 + \frac{m\omega_c}{4}\theta\right) p_\mu \qquad \mu = x, y. \tag{9}$$

This problem has been analysed without the confining potential and at noncommutative level on the torus [18]. Note that when θ vanishes, the standard Hamiltonian can be recovered.

To close this section, we mention that the Hamiltonian equation (4) has been considered on the noncommutative space [19], where the relation

$$[p_x, p_y] = \mathrm{i}B \tag{10}$$

and the convention ($e = 1, c = 1$) are used, which is not the case for our analysis. However, we can find identical results concerning the noncommutative Hamiltonian formalism if we make a redefinition of the magnetic field, such that

$$B_P = -B_J\left(1 + \frac{\theta}{4}B_J\right) \tag{11}$$

where B_P is the magnetic field used by Polychronakos *et al* and B_J is the one appearing in our formulas[2].

3. Eigenstates and eigenvalues of H_θ

We adopt two methods to obtain the energy spectrum and the eigenstates of H_θ. The first one utilizes Weyl–Heisenberg symmetries and the last one is related to the stationary Schrödinger equation.

3.1. Algebraic method

It is possible to write the noncommutative Hamiltonian as the sum of two independent harmonic oscillator Hamiltonians $\tilde{H}_0$ plus the angular momentum operator on z-direction L_z. Therefore, we have

$$H_\theta = \tilde{H}_0 + \frac{\tilde{\omega}_c}{2}L_z \tag{12}$$

where $\tilde{H}_0$ and L_z are given by

$$\begin{aligned}\tilde{H}_0 &= \frac{1}{2m}\left(\hat{p}_x^2 + \frac{1}{8}m\omega^2x^2\right) + \frac{1}{2m}\left(\hat{p}_y^2 + \frac{1}{8}m\omega^2y^2\right)\\ L_z &= xp_y - yp_x.\end{aligned} \tag{13}$$

Here $\omega_c = eB/mc$ is the cyclotron frequency, $\omega = \sqrt{\omega_c^2 + 4\omega_0^2}$ and

$$\hat{p}_\mu^2 = \left(1 + \frac{m\omega_c}{2}\theta + \left(\frac{m\omega}{4}\theta\right)^2\right)p_\mu^2 \qquad \tilde{\omega}_c = \omega_c\left(1 + \left(\frac{\omega_c}{4} - \frac{\omega_0^2}{\omega_c}\right)m\theta\right). \tag{14}$$

We want to express H_θ in terms of creation and annihilation operators. For that, we introduce the following operators in the complex plane $(z, \bar{z})$:

$$\begin{aligned}\tilde{a}_d &= \frac{1}{2}\left(\tilde{\xi}\bar{z} + \frac{\mathrm{i}}{2\hbar\tilde{\xi}}p_z\right) \qquad & \tilde{a}_d^\dagger &= \frac{1}{2}\left(\tilde{\xi}z - \frac{\mathrm{i}}{2\hbar\tilde{\xi}}p_{\bar{z}}\right)\\ \tilde{a}_g &= \frac{1}{2}\left(\tilde{\xi}z + \frac{\mathrm{i}}{2\hbar\tilde{\xi}}p_{\bar{z}}\right) \qquad & \tilde{a}_g^\dagger &= \frac{1}{2}\left(\tilde{\xi}\bar{z} - \frac{\mathrm{i}}{2\hbar\tilde{\xi}}p_z\right)\end{aligned} \tag{15}$$

where $\tilde{\xi}$ is a θ-function, such that

$$\tilde{\xi} = \sqrt[4]{\frac{(m\omega/2\hbar)^2}{1 + \frac{m\omega_c}{2}\theta + (\frac{m\omega}{4}\theta)^2}}. \tag{16}$$

[2] I am grateful to Polychronakos for pointing out equation (11) on 31 May 2001.

It is easy to show that

$$[\tilde{a}_d, \tilde{a}_d^\dagger] = 1 = [\tilde{a}_g, \tilde{a}_g^\dagger] \tag{17}$$

and other commutators vanish. Consequently, $\tilde{H}_0$ and L_z take the new forms

$$\tilde{H}_0 = \frac{\hbar\tilde{\omega}}{2}\,(\tilde{N}_d + \tilde{N}_g + 1) \qquad L_z = \hbar(\tilde{N}_d - \tilde{N}_g) \tag{18}$$

where $\tilde{N}_d = \tilde{a}_d^\dagger \tilde{a}_d$, $\tilde{N}_g = \tilde{a}_g^\dagger \tilde{a}_g$ are the number operators and $\tilde{\omega}$ is θ-dependent:

$$\tilde{\omega} = \omega\sqrt{1 + \frac{m\omega_c}{2}\theta + \left(\frac{m\omega}{4}\theta\right)^2}. \tag{19}$$

Actually, we have the following expression for the noncommutative Hamiltonian:

$$H_\theta = \frac{\hbar\tilde{\omega}}{2}\,(\tilde{N}_d + \tilde{N}_g + 1) + \frac{\hbar\tilde{\omega}_c}{2}\,(\tilde{N}_d - \tilde{N}_g). \tag{20}$$

The latter can be arranged as follows:

$$H_\theta = \frac{\hbar}{2}\,(\tilde{N}_d\tilde{\omega}_+ + \tilde{N}_g\tilde{\omega}_- + \tilde{\omega}) \tag{21}$$

and we have

$$\tilde{\omega}_\pm = \omega\sqrt{1 + \frac{m\omega_c}{2}\theta + \left(\frac{m\omega}{4}\theta\right)^2} \pm \omega_c\left(1 + \left(\frac{\omega_c}{4} - \frac{\omega_0^2}{\omega_c}\right)m\theta\right) = \tilde{\omega} \pm \tilde{\omega}_c. \tag{22}$$

We derive immediately the energy spectrum from the relation

$$\tilde{H}_\theta \mid \tilde{n}_d, \tilde{n}_g\rangle = E_{\tilde{n}_d\tilde{n}_g} \mid \tilde{n}_d, \tilde{n}_g\rangle \tag{23}$$

which leads to

$$E_{\tilde{n}_d\tilde{n}_g} = \frac{\hbar}{2}(\tilde{n}_d\tilde{\omega}_+ + \tilde{n}_g\tilde{\omega}_- + \tilde{\omega}) \tag{24}$$

where $\tilde{n}_d$ and $\tilde{n}_g$ are non-negative integers. The corresponding eigenstates are tensor products of single Fock oscillator states:

$$\mid \tilde{n}_d, \tilde{n}_g\rangle = \frac{1}{\sqrt{\tilde{n}_d!\tilde{n}_g!}}(\tilde{a}_d^\dagger)^{\tilde{n}_d}(\tilde{a}_g^\dagger)^{\tilde{n}_g} \mid \tilde{0}, \tilde{0}\rangle \tag{25}$$

where $\mid \tilde{0}, \tilde{0}\rangle$ is the vacuum of H_θ. Note that if we use equation (11), we recover the results obtained in [19] for the Hamiltonian equation (4) in noncommutative space.

3.2. *Analytical method*

To obtain the analytical solutions of the present problem, we introduce the polar coordinates $(x, y) = (r\sin\varphi, r\cos\varphi)$, with $0 < r < \infty$ and $0 \leqslant \varphi \leqslant \pi$. In this case, the stationary Schrödinger equation can be written as follows:

$$\left(-\frac{\hbar^2}{2m}\left(1 + \frac{m\omega_c}{2}\theta\right) + \left(\frac{m\omega}{4}\theta\right)^2\right)\left(\partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\varphi^2\right) - \mathrm{i}\frac{\hbar\tilde{\omega}_c}{2}\partial_\varphi + \frac{m}{8}\omega^2 r^2\right)\Psi_\theta(r,\varphi)$$
$$= E_\theta\Psi_\theta(r,\varphi). \tag{26}$$

Note that H_θ and L_z commute. Therefore, following the fundamental principle of quantum mechanics, these operators have a common basis of eigenvectors. Then, by choosing these eigenfunctions as $\Psi_\theta(r,\varphi) = R_\theta(r)\mathrm{e}^{\mathrm{i}\alpha\varphi}$, we can show that equation (26) yields

$$\left(\partial_r^2 + \frac{1}{r}\partial_r - \frac{\alpha^2}{r^2}\right)R_\theta(r) - \left(\tilde{k}^2 - \tilde{\zeta}_2 r^2\right)R_\theta(r) = 0 \tag{27}$$

where

$$\tilde{k}^2 = \frac{E_\theta - \frac{\hbar\omega_c}{2}\left(1 + \left(\frac{\omega_c}{4} - \frac{\omega_0^2}{\omega_c}\right) m\theta\right)\alpha}{\frac{\hbar^2}{2}\left(1 + \frac{m\omega_c}{2}\theta + \left(\frac{m\omega}{4}\theta\right)^2\right)} \qquad \tilde{\zeta}^2 = \frac{(m\omega/2\hbar)^2}{1 + \frac{m\omega_c}{2}\theta + \left(\frac{m\omega}{4}\theta\right)^2}. \tag{28}$$

By straighforward computation we show that

$$R_\theta(r) = r^{|\alpha|} \exp\left(-\frac{\tilde{\zeta} r^2}{2}\right) L_\theta(r) \tag{29}$$

is a solution of the above equation, where the $L_\theta(r)$ are the Laguerre polynomials obeying

$$\partial_r^2 L_\theta + \left(\frac{2|\alpha| + 1}{r} - 2\alpha r\right) \partial_r L_\theta - (2\tilde{\zeta}(|\alpha| + 1) - \tilde{k}^2) L_\theta = 0. \tag{30}$$

Therefore, we can obtain the explicit eigenstates of H_θ as

$$\Psi_\theta(r,\varphi) = \Psi_{n,\alpha,\theta}(r,\varphi) = (-1)^n \sqrt{\frac{\tilde{\zeta}}{\pi}} \sqrt{\frac{n!}{(n + |\alpha|)!}} \exp\left(-\frac{\tilde{\zeta} r^2}{2}\right) \left(\sqrt{\tilde{\zeta}} r\right)^{|\alpha|} L_{n,\theta}^{(|\alpha|)}\left(\tilde{\zeta} r^2\right) \mathrm{e}^{\mathrm{i}\alpha\varphi} \tag{31}$$

where

$$L_{n,\theta}^{(|\alpha|)}\left(\tilde{\zeta} r^2\right) = \sum_{m=0}^{n} (-1)^m \binom{n + |\alpha|}{n - m} \frac{\left(\tilde{\zeta} r^2\right)^m}{m!} \qquad n = 0, 1, 2, \ldots$$

is the principal quantum number and $\alpha = 0, \pm 1, \pm 2, \ldots$ is the angular momentum quantum number. The corresponding spectrum is given by

$$E_{n\alpha,\theta} = \frac{\hbar\tilde{\omega}}{2}\left(n + \frac{|\alpha| + 1}{2}\right) + \frac{\hbar\tilde{\omega}_c}{2}\alpha. \tag{32}$$

Since $L_{n,\theta}^{(|\alpha|)}(0) = \frac{(n+\alpha)(n+\alpha-1)\cdots(\alpha+1)}{n!}$, from equation (31) we observe immediately that $\Psi_{n,\alpha,\theta}(0) = 0$ and also becomes zero when r goes to infinity. Returning now to the algebraic method, we can see that n and α are connected to $\tilde{n}_d$ and $\tilde{n}_g$ by

$$\tilde{n}_d = n + \tfrac{1}{2}(|\alpha| + \alpha) \quad \text{and} \quad \tilde{n}_g = n + \tfrac{1}{2}(|\alpha| - \alpha).$$

Note that $\Psi_{n,\alpha,\theta}(r,\varphi) = \langle r,\varphi \mid n,\alpha\rangle = \langle r,\varphi \mid \tilde{n}_d, \tilde{n}_g\rangle$.

4. Degeneracy of Landau levels

As in the commutative case [13], we can give a realization of certain algebras, in particular $su(2)$ and $su(1,1)$, in terms of the creation and annihilation operators defined before. We can also study some particular cases in which the magnetic field takes some limiting values.

4.1. Algebras su(2) and su(1, 1)

We start with the former one. The algebra generators can be built as

$$\tilde{S}_+ = \tilde{a}_d^\dagger \tilde{a}_g \qquad \tilde{S}_- = \tilde{a}_g^\dagger \tilde{a}_d \qquad \tilde{S}_z = \frac{\tilde{N}_d - \tilde{N}_g}{2} = \frac{L_z}{2\hbar}. \tag{33}$$

It easy to show that these generators verify the following commutation relations:

$$[\tilde{S}_+, \tilde{S}_-] = 2\tilde{S}_z \qquad [\tilde{S}_z, \tilde{S}_\pm] = \pm\tilde{S}_\pm. \tag{34}$$

Subsequently, we can also define the invariant Casimir operator in terms of $su(2)$ generators,

$$\tilde{\mathcal{C}} = \frac{1}{2}(\tilde{S}_+\tilde{S}_- + \tilde{S}_-\tilde{S}_+) + \tilde{S}_z^2 = \left(\frac{\tilde{N}_d + \tilde{N}_g}{2}\right)\left(\frac{\tilde{N}_d + \tilde{N}_g}{2} + 1\right). \tag{35}$$

We prove that H_θ is not invariant under this algebra. As in the commutative case, for a given value $\gamma = (\tilde{n}_d + \tilde{n}_g)/2$, there exists a $(2\gamma + 1)$-dimensional UIR of $su(2)$ in which the operator $\tilde{S}_z$ has its spectral values in the range $-\gamma \leqslant \rho = (\tilde{n}_d - \tilde{n}_g)/2 \leqslant \gamma$.

Following the same idea the other algebra can be realized as follows:

$$\tilde{T}_+ = \tilde{a}_d^\dagger \tilde{a}_g^\dagger \qquad \tilde{T}_- = \tilde{a}_d \tilde{a}_g \qquad \tilde{T}_0 = \frac{1}{2}(\tilde{N}_d + \tilde{N}_g + 1) = \frac{\tilde{H}_0}{\hbar\tilde{\omega}}. \tag{36}$$

Then we reproduce the commutation relations generating the $su(1, 1)$ algebra:

$$[\tilde{T}_+, \tilde{T}_-] = -2\tilde{T}_0 \qquad [\tilde{T}_0, \tilde{T}_\pm] = \pm\tilde{T}_\pm. \tag{37}$$

Furthermore, its Casimir operator is given by

$$\tilde{\mathcal{D}} = \frac{1}{2}(\tilde{T}_+\tilde{T}_- + \tilde{T}_-\tilde{T}_+) - \tilde{T}_0^2 = -\left(\frac{\tilde{N}_d - \tilde{N}_g}{2} + \frac{1}{2}\right)\left(\frac{\tilde{N}_d - \tilde{N}_g}{2} - \frac{1}{2}\right) = -\frac{1}{4}\left(\frac{L_z}{\hbar}^2 - 1\right). \tag{38}$$

This algebra also is not a symmetry of the noncommutative Hamiltonian. Note that, when $\tilde{n}_d \geqslant \tilde{n}_g$, for a given value $\eta = (\tilde{n}_d - \tilde{n}_d + 1)/2 \geqslant 1/2$, there exists a UIR of $su(1, 1)$ in the discrete series, in which the operator $\tilde{T}_0$ has its spectral values in the infinite range $\eta, \eta + 1, \eta + 2, \ldots$. However, when $\tilde{n}_d \leqslant \tilde{n}_g$, for a given value $\vartheta = (-\tilde{n}_d + \tilde{n}_g + 1)/2 \geqslant 1/2$, there also exists a UIR of $su(1, 1)$ in which the spectral value of the operator $\tilde{T}_0$ runs in the infinite range $\vartheta, \vartheta + 1, \vartheta + 2, \ldots$.

4.2. Magnetic field limits

Let us examine some particalur cases of the magnetic field: weak-field and strong-field limits. We begin by arranging the energy spectrum as follows:

$$\tilde{E}_{n\alpha} = \frac{\hbar\tilde{\omega}}{2}\,\gamma + \frac{\hbar\tilde{\omega}_c}{2}\,\rho + \frac{\hbar\tilde{\omega}}{2}. \tag{39}$$

Weak field case. Suppose that $\omega_c \ll \omega_0$, then the above equation can be approximated by

$$\tilde{E}_{n\alpha} \approx \hbar\omega_0\sqrt{1 + \left(\frac{m\omega_0}{2}\theta\right)^2}\,(2\gamma + 1) - \frac{\hbar m\omega_0^2}{2}\theta\rho \equiv E_{\gamma,\rho} \tag{40}$$

which tells us that there is no degeneracy of Landau levels. This effect is due to the presence of the θ-term in the energy spectrum equation (40). The latter shows a difference with the commutative case, where we have pointed out [13] that $su(2)$ is behind the degeneracy of Landau levels at weak field.

Strong field case. In the limit of strong magnetic field $\omega_c \gg \omega_0$, we have

$$E_{\tilde{n}_d\tilde{n}_g} \approx \hbar\omega_c\left(1 + \frac{m\omega_c}{4}\theta\right)\left(\tilde{n}_d + \frac{1}{2}\right). \tag{41}$$

As in the commutative case by redefining ω_c we get harmonic oscillator and it is still true that for a given value of $\tilde{n}_d$, we have an infinite degeneracy labelled by $\tilde{n}_g$ or by $\alpha = \tilde{n}_d - \tilde{n}_g \leqslant \tilde{n}_d$. The quantum number $\tilde{n}_d$ corresponds to the Landau level index (as well as n for negative α). One can reinterpret it in terms of $su(1, 1)$ symmetry by noting that, for a given value of $\alpha \leqslant 0$, the energy eigenstates are ladder states for the discrete series representation labelled by $\vartheta = -\alpha/2 + 1/2$.

Generic intermediate case. We distinguish two cases: for $\tilde{\omega}_+/\tilde{\omega}_- \notin \mathbb{Q}$, what we can do is just write the energy spectrum in the form

$$\mathcal{E}_{\tilde{n}_d\tilde{n}_g} \equiv \frac{E_{\tilde{n}_d\tilde{n}_g}}{\hbar\tilde{\omega}_-} - \frac{\tilde{\omega}}{2\tilde{\omega}_-} = \frac{\tilde{\omega}_+}{\tilde{\omega}_-}\tilde{n}_d + \tilde{n}_g \tag{42}$$

otherwise there is no information about degeneracy. For $\tilde{\omega}_+/\tilde{\omega}_- = p/q \in \mathbb{Q}$, the latter is possible:

$$E_{\tilde{n}_d\tilde{n}_g} = E_{\tilde{n}'_d\tilde{n}'_g} \quad \text{iff} \quad \frac{p}{q} = -\frac{\tilde{n}_g - \tilde{n}'_g}{\tilde{n}_d - \tilde{n}'_d}. \tag{43}$$

Conclusion. The above analysis leads us to conclude that the introduction of the noncommutative parameter can solve some problems. For instance, the degeneracy of the Landau levels is lifted via the θ-term for a weak magnetic field. Indeed, for any non-zero θ-value, the term $\frac{\hbar m\omega_0^2}{2}\theta\rho$ is present in equation (40), which means that for any given eigenvalue $E_{\lambda,\rho}$ there is only one eigenfunction parametrized by the same integers λ and ρ. However for $\theta = 0$, we recover the harmonic oscillator for two-dimensional Landau problem, which is a degenerate system.

5. Thermodynamical potential

We make the assumption that the total number $\langle\tilde{N}_e\rangle$ of electrons is large enough so that the difference between a grand canonical ensemble and a canonical one is not of importance [12, 13]. Then, the thermodynamical potential can be written as follows:

$$\Omega_\theta = -\frac{1}{\beta}\text{Tr}\ \log\left(1 + \mathrm{e}^{-\beta(H_\theta-\mu)}\right) \tag{44}$$

with $\beta = 1/(k_B T)$. Evaluating the trace on the derived eigenstates, we obtain

$$\Omega_\theta = \sum_{\tilde{n}_d,\tilde{n}_g}^{\infty} \log\left(1 + \mathrm{e}^{-\beta(\frac{\hbar}{2}(\tilde{\omega}_+\tilde{n}_d+\tilde{\omega}_-\tilde{n}_g+\tilde{\omega})-\mu)}\right). \tag{45}$$

By definition, the magnetic moment M_θ is

$$M_\theta = -\left(\frac{\partial\Omega_\theta}{\partial B}\right)_\mu \tag{46}$$

and the average number of electrons is given by

$$\langle\tilde{N}_e\rangle = -\partial_\mu\Omega_\theta. \tag{47}$$

On the other hand, it is not easy to manipulate directly equation (45) and subsequently equations (46) and (47). Basically, we need some tools to do that; this is the reason why we introduce coherent states [20, 21]. Then, before investigating the thermodynamical potential, we start with constructing the coherent states. Note that this construction is, more or less, the same as in the standard case.

5.1. Coherent states

Using standard methods, the coherent states for the present system can be constructed as follows:

$$|\ \tilde{z}_d, \tilde{z}_g\rangle = \exp\left[-\tfrac{1}{2}(|\tilde{z}_d|^2 + |\tilde{z}_g|^2)\right]\mathrm{e}^{\tilde{z}_d\tilde{a}_d^\dagger+\tilde{z}_g\tilde{a}_g^\dagger}\ |\ \tilde{0},\tilde{0}\rangle. \tag{48}$$

It easy to observe that

$$\tilde{a}_d \mid \tilde{z}_d, \tilde{z}_g\rangle = \tilde{z}_d \mid \tilde{z}_d, \tilde{z}_g\rangle \qquad \tilde{a}_g \mid \tilde{z}_d, \tilde{z}_g\rangle = \tilde{z}_g \mid \tilde{z}_d, \tilde{z}_g\rangle. \tag{49}$$

We cite some interesting properties, which will be useful in the next. The first one is the action identity

$$\check{H}_\theta(\tilde{z}_d, \tilde{z}_g) \equiv \langle \tilde{z}_d, \tilde{z}_g \mid H_\theta \mid \tilde{z}_d, \tilde{z}_g\rangle = \frac{\hbar}{2}\Big(\tilde{\omega}_+ |\tilde{z}_d|^2 + \tilde{\omega}_- |\tilde{z}_g|^2 + \tilde{\omega}\Big). \tag{50}$$

In the literature, the function $\check{H}_\theta(\tilde{z}_d, \tilde{z}_g)$ is known as the lower (resp. contravariant) symbol of the operator H_θ [23, 26]. It will play an important role in the present context. The second one is the resolution of the unity:

$$I = \frac{1}{\pi^2}\int_{\mathbb{C}^2} \mid \tilde{z}_d, \tilde{z}_g\rangle\langle \tilde{z}_d, \tilde{z}_g \mid \mathrm{d}^2\tilde{z}_d\, \mathrm{d}^2\tilde{z}_g. \tag{51}$$

The last property is also crucial in our context. Indeed, for any observable A with suitable operator properties (trace-class, etc), there exists a unique upper (or covariant) symbol $\hat{A}(\tilde{z}_d, \tilde{z}_g)$ defined by

$$A = \frac{1}{\pi^2}\int_{\mathbb{C}^2} \hat{A}(\tilde{z}_d, \tilde{z}_g) \mid \tilde{z}_d, \tilde{z}_g\rangle\langle \tilde{z}_d, \tilde{z}_g \mid \mathrm{d}^2\tilde{z}_d\, \mathrm{d}^2\tilde{z}_g. \tag{52}$$

It is easy to see that the upper symbols for number operators are

$$\hat{\hat{N}}_d(\tilde{z}_d, \tilde{z}_g) = |\tilde{z}_d|^2 - 1 \qquad \hat{\hat{N}}_g(\tilde{z}_d, \tilde{z}_g) = |\tilde{z}_d|^2 - 1 \tag{53}$$

which imply the following one for the noncommutative Hamiltonian

$$\hat{H}_\theta(\tilde{z}_d, \tilde{z}_g) = \frac{\hbar}{2}\Big(\tilde{\omega}_+ |\tilde{z}_d|^2 + \tilde{\omega}_- |\tilde{z}_g|^2 - \tilde{\omega}\Big). \tag{54}$$

Note that there is another useful trace identity for a trace-class observable A, such that

$$\mathrm{Tr}\, A = \frac{1}{\pi^2}\int_{\mathbb{C}^2} \check{A}(\tilde{z}_d, \tilde{z}_g)\, \mathrm{d}^2\tilde{z}_d\, \mathrm{d}^2\tilde{z}_g = \frac{1}{\pi^2}\int_{\mathbb{C}^2} \hat{A}(\tilde{z}_d, \tilde{z}_g)\, \mathrm{d}^2\tilde{z}_d\, \mathrm{d}^2\tilde{z}_g \tag{55}$$

where we have $\check{A}(\tilde{z}_d, \tilde{z}_g) \equiv \langle \tilde{z}_d, \tilde{z}_g \mid A \mid \tilde{z}_d, \tilde{z}_g\rangle$.

5.2. Berezin–Lieb inequalities

Let us observe that $\log(1 + \mathrm{e}^{-\beta(H_\theta - \mu)})$ is a convex function of the positive Hamiltonian H_θ. Then, the Berezin–Lieb inequalities can be applied to study the quasi-classical behaviour of the thermodynamical potential. For any convex function $g(A)$ of the observable A it is possible to write [23, 26]

$$\frac{1}{\pi^2}\int_{\mathbb{C}^2} g(\check{A})\, \mathrm{d}^2\tilde{z}_d\, \mathrm{d}^2\tilde{z}_g \leqslant \mathrm{Tr} g(A) \leqslant \frac{1}{\pi^2}\int_{\mathbb{C}^2} g(\hat{A})\, \mathrm{d}^2\tilde{z}_d\, \mathrm{d}^2\tilde{z}_g. \tag{56}$$

This formula can be used for evaluating the (concave) thermodynamical potential. Then, we have

$$-\frac{1}{\beta\pi^2}\int_{\mathbb{C}^2} \log\left(1 + \mathrm{e}^{-\beta(\hat{H}_\theta - \mu)}\right) \mathrm{d}^2\tilde{z}_d\, \mathrm{d}^2\tilde{z}_g \leqslant \Omega_\theta \leqslant -\frac{1}{\beta\pi^2}\int_{\mathbb{C}^2} \log\left(1 + \mathrm{e}^{-\beta(\check{H}_\theta - \mu)}\right) \mathrm{d}^2\tilde{z}_d\, \mathrm{d}^2\tilde{z}_g. \tag{57}$$

Using equations (50) and (54) and performing the angular integrations, we get

$$\begin{aligned} -\frac{1}{\beta}\int_0^\infty \mathrm{d}\tilde{u}_d \int_0^\infty \mathrm{d}\tilde{u}_g \log\left(1 + \mathrm{e}^{-\beta(\frac{\hbar}{2}(\tilde{\omega}_+\tilde{u}_d + \tilde{\omega}_-\tilde{u}_g - \tilde{\omega}) - \mu)}\right) \leqslant \Omega_\theta \\ \Omega_\theta \leqslant -\frac{1}{\beta}\int_0^\infty \mathrm{d}\tilde{u}_d \int_0^\infty \mathrm{d}\tilde{u}_g \log\left(1 + \mathrm{e}^{-\beta(\frac{\hbar}{2}(\tilde{\omega}_+\tilde{u}_d + \tilde{\omega}_-\tilde{u}_g + \tilde{\omega}) - \mu)}\right) \end{aligned} \tag{58}$$

where $\tilde{u}_d = |\tilde{z}_d|^2$ and $\tilde{u}_g = |\tilde{z}_g|^2$. In order to calculate the last integrals, we put $\tilde{u} = \frac{\beta\hbar}{2}(\tilde{\omega}_+\tilde{u}_d + \tilde{\omega}_-\tilde{u}_g)$, $\tilde{v} = \frac{\beta\hbar}{2}\tilde{\omega}_+\tilde{u}_d$, then performing an integration by parts, and introducing the control parameters $\tilde{\kappa}_\pm = \exp(\beta(\mu \pm \hbar\tilde{\omega}/2))$, we obtain

$$\phi(\tilde{\kappa}_+) \leqslant \Omega_\theta \leqslant \phi(\tilde{\kappa}_-) \tag{59}$$

where $\phi(\tilde{\kappa})$ takes the form

$$\begin{aligned}\phi(\tilde{\kappa}) &= -\frac{2\tilde{\kappa}}{\beta(\beta\hbar)^2\tilde{\omega}_+\tilde{\omega}_-}\int_0^\infty \frac{\tilde{u}^2\mathrm{e}^{-\tilde{u}}}{1+\tilde{\kappa}\mathrm{e}^{-\tilde{u}}}\,\mathrm{d}\tilde{u} \\ &= \begin{cases}\frac{4}{\beta(\beta\hbar)^2\tilde{\omega}_+\tilde{\omega}_-}\,\tilde{F}_3(-\tilde{\kappa}) & \text{for} \quad \tilde{\kappa} \leqslant 1 \\ \frac{4}{\beta(\beta\hbar)^2\tilde{\omega}_+\tilde{\omega}_-}\left[-\frac{(\log\tilde{\kappa})^3}{6} - \frac{\pi^2\log\tilde{\kappa}}{6} + \tilde{F}_3(-\tilde{\kappa}^{-1})\right] & \text{for} \quad \tilde{\kappa} > 1\end{cases}\end{aligned} \tag{60}$$

and the function $\tilde{F}_s$ is Riemann–Fermi–Dirac type, such that

$$\tilde{F}_s(\tilde{z}) = \sum_{n=1}^{\infty}\frac{\tilde{z}^n}{n^s}. \tag{61}$$

Since we have a term $\tilde{\omega}_+\tilde{\omega}_-$ in the denominator of equation (60), we note that

$$\tilde{\omega}_+\tilde{\omega}_- = \omega_0^2(2 + m\omega_c\theta)^2. \tag{62}$$

We observe that equation (60) shows a singularity at a critical point. So we are now forced to distinguish two different cases. The first one, $m\omega_c\theta \neq -2$, is equivalent actually to $\frac{eB\theta}{c} \neq 2$. Then we have $m\omega_c\theta >$ or < -2, since there is a square, we can only discuss the global case. Second one is a critical point $m\omega_c\theta = -2$ where equation (60) diverges. Remembering that by using equation (11), we find that our cases coincide with those noted in [19], namely $B\theta \neq 1$ and $B\theta = 1$. In this section, we assume that the former case holds in further analysis. However, the latter case will deal with the last section.

Let us examine equation (60) in different limits of temperature and by putting the condition: $m\omega_c\theta \neq -2$. In other words, we want to derive the thermodynamical potential and the related physical quantities at high and low temperatures at noncommutative level and compare with the standard case.

High-temperature limit. In this case we make the assumption $|\mu \pm \hbar\tilde{\omega}/2| \gg \beta$ and get $\tilde{\kappa}_\pm \approx 1$. Therefore using equations (59) and (60), Ω_θ can be approximated by

$$\Omega_\theta \approx \frac{4}{\beta^3\hbar^2}\,\frac{F_3(-1)}{\tilde{\omega}_+\tilde{\omega}_-} \tag{63}$$

where $F_3(-1) = -0.901\,543$. In terms of θ we have

$$\Omega_\theta \approx -0.901\,543 \times 4\,\frac{\left(\frac{1}{\beta\hbar\omega_0}\right)^2}{\beta(2 + m\omega_c\theta)^2}. \tag{64}$$

We remark from the last formula that $\partial_\mu\Omega_\theta = 0$, namely there is no exchange of electrons. This means that at high temperature, the present system can be described as a canonical ensemble. However, the magnetization and susceptibility can be evaluated in this case. We get for M_θ

$$M_\theta = 0.901\,543 \times 8\left(\frac{1}{\beta\hbar\omega_0}\right)^2\frac{e\theta/\beta c}{(2 + m\omega_c\theta)^3} \tag{65}$$

and remembering the relation $\chi_\theta = \frac{\partial M_\theta}{\partial B}$, we obtain for the susceptibility

$$\chi_\theta = -0.901\,543 \times 24\,\frac{1}{\beta^3}\left(\frac{e\theta}{\hbar c\omega_0}\right)^2\frac{1}{(2 + m\omega_c\theta)^4}. \tag{66}$$

Let us examine some particular cases of the last equation. For a zero magnetic field, we find

$$\chi_\theta = -0.901\ 543 \times \frac{3}{2}\,\frac{1}{\beta^3}\left(\frac{e\theta}{\hbar c\omega_0}\right)^2. \tag{67}$$

This latter shows that χ_θ is θ-dependent. Therefore, we have Landau diamagnetism since θ is a real value. However, when θ vanishes there is no magnetic behaviour. This means that

$$\chi_{\theta=0} = 0 \tag{68}$$

which is compatible with the standard case. It is interesting to note that at high temperature the system presents a magnetic behaviour in terms of θ and it is the canonical one. This effect does not appear in the commutative case. This is one of the original results derived in this paper.

Low-temperature limit. Let us consider another interesting case, namely $\mu \gg \hbar\tilde{\omega}/2$ and $\mu \ll \beta$. In this situation, $\phi(\tilde{\kappa})$ can be expressed as

$$\phi(\tilde{\kappa}_\pm) = \tilde{A} \mp \frac{\tilde{\Delta}}{2} + \tilde{S}_\pm \tag{69}$$

and we have

$$\begin{aligned}
\tilde{A} &= -2\mu\,\frac{\frac{1}{3}\left(\frac{\mu}{\hbar\omega_0}\right)^2 + \frac{1}{4}\left(\frac{\tilde{\omega}}{\omega_0}\right)^2 + \frac{\pi^2}{3}\left(\frac{1}{\beta\hbar\omega_0}\right)^2}{(2+m\omega_c\theta)^2}\\
\frac{\tilde{\Delta}}{2} &= 2\hbar\tilde{\omega}\frac{\frac{1}{2}\left(\frac{\mu}{\hbar\omega_0}\right)^2 + \frac{1}{24}\left(\frac{\tilde{\omega}}{\omega_0}\right)^2 + \frac{\pi^2}{6}\left(\frac{1}{\beta\hbar\omega_0}\right)^2}{(2+m\omega_c\theta)^2}\\
\tilde{S}_\pm &= \left(\frac{1}{\beta\hbar\omega_0}\right)^2\,\frac{4}{\beta(2+m\omega_c\theta)^2}\,F_3(-\exp[-\beta(\mu\pm\hbar\tilde{\omega}/2)]).
\end{aligned} \tag{70}$$

At low temperature, $\tilde{S}_\pm$ can be approximated by the following relation:

$$\tilde{S}_0 = \left(\frac{1}{\beta\hbar\omega_0}\right)^2\,F_3(-\mathrm{e}^{-\beta\mu})\,\frac{4}{\beta(2+m\omega_c\theta)^2} \tag{71}$$

and we can see immediately that the equation

$$\frac{\tilde{\Delta}}{|\tilde{A}+\tilde{S}_0|} = \frac{\hbar\tilde{\omega}}{\mu}\left[\frac{3+\pi^2\left(\frac{1}{\beta\mu}\right)^2 + \frac{1}{4}\left(\frac{\hbar\tilde{\omega}}{\mu}\right)^2}{1+\pi^2\left(\frac{1}{\beta\mu}\right)^2 + \frac{3}{4}\left(\frac{\hbar\tilde{\omega}}{\mu}\right)^2 - \left(\frac{1}{\beta\mu}\right)^3 F_3(-\mathrm{e}^{-\beta\mu})}\right] \tag{72}$$

tends to zero. Therefore, the thermodynamical potential can be written as follows:

$$\begin{aligned}
\Omega_\theta = -\frac{2\mu}{(2+m\omega_c\theta)^2}&\left[\frac{1}{3}\left(\frac{\mu}{\hbar\omega_0}\right)^2 + \frac{1}{4}\left(\frac{\tilde{\omega}}{\omega_0}\right)^2 + \frac{\pi^2}{3}\left(\frac{1}{\beta\hbar\omega_0}\right)^2\right.\\
&\left.-\frac{2}{\beta\mu}\left(\frac{1}{\beta\hbar\omega_0}\right)^2 F_3(-\mathrm{e}^{-\beta\mu})\right].
\end{aligned} \tag{73}$$

In this quasiclassical regime, the average number of electrons is

$$\begin{aligned}
\langle\tilde{N}_e\rangle = 4\left(\frac{\mu/\hbar\omega_0}{2+m\omega_c\theta}\right)^2&\left[\frac{1}{2} + \frac{1}{8}\left(\frac{\hbar\tilde{\omega}}{\mu}\right)^2 + \frac{\pi^2}{6}\left(\frac{1}{\beta\mu}\right)^2\right.\\
&\left.+\left(\frac{\omega_0}{\tilde{\omega}}\right)^2(2+m\omega_c\theta)^2\left(\frac{1}{\beta\mu}\right)^2 F_2(-\mathrm{e}^{-\mu\beta})\right]
\end{aligned} \tag{74}$$

which can be estimated as

$$\langle \tilde{N}_e \rangle \approx 2 \left(\frac{\mu}{\hbar\omega_0}\right)^2 \frac{1}{(2+m\omega_c\theta)^2}. \tag{75}$$

Note that the average number of electrons at low T in the commutative case can be recovered just by switching off one of the parameters B or θ. By using the definition of magnetic moment, we obtain

$$M_\theta = -4\frac{e\mu\theta/c}{(2+m\omega_c\theta)^3}\left[\frac{1}{3}\left(\frac{\mu}{\hbar\omega_0}\right)^2 + \frac{1}{4}\left(\frac{\tilde{\omega}}{\omega_0}\right)^2 + \frac{\pi^2}{3}\left(\frac{1}{\beta\hbar\omega_0}\right)^2 - \frac{2}{\beta\mu}\left(\frac{1}{\beta\hbar\omega_0}\right)^2 \right.$$
$$\left. \times F_3(-\mathrm{e}^{-\beta\mu})\right] + \frac{e\mu/mc}{(2+m\omega_c\theta)^2}\left[\frac{\omega_c}{\omega_0^2} + \frac{m\theta}{4\omega_0^2}\left(2\omega_c^2+\omega^2\right) + 2\frac{\omega_c}{\omega_0^2}\left(\frac{m\omega\theta}{4}\right)^2\right]. \tag{76}$$

Therefore, the susceptibility takes the following form:

$$\chi_\theta = 12\mu\frac{(e\theta/c)^2}{(2+m\omega_c\theta)^4}\left[\frac{1}{3}\left(\frac{\mu}{\hbar\omega_0}\right)^2 + \frac{1}{4}\left(\frac{\tilde{\omega}}{\omega_0}\right)^2 + \frac{\pi^2}{3}\left(\frac{1}{\beta\hbar\omega_0}\right)^2 \right.$$
$$\left. - \frac{2}{\beta\mu}\left(\frac{1}{\beta\hbar\omega_0}\right)^2 F_3(-\mathrm{e}^{-\beta\mu})\right]$$
$$- 4\mu\left(\frac{e}{mc}\right)^2 \frac{m\theta}{(2+m\omega_c\theta)^3}\left[\frac{\omega_c}{\omega_0^2} + \frac{m\theta}{4\omega_0^2}\left(2\omega_c^2+\omega^2\right) + 2\frac{\omega_c}{\omega_0^2}\left(\frac{m\omega\theta}{4}\right)^2\right]$$
$$+ \left(\frac{e}{mc\omega_0}\right)^2 \frac{\mu}{(2+m\omega_c\theta)^2}\left[1 + 2m\omega_c\theta + 6\left(\frac{m\omega\theta}{4}\right)^2\right]. \tag{77}$$

For zero magnetic field, we find

$$\chi_\theta = \chi_p\left[1 + (m\omega_0\theta)^2\left\{1 + \left(\frac{\mu}{\hbar\omega_0}\right)^2 + 3\left(\frac{m\omega_0\theta}{2}\right)^2 \right.\right.$$
$$\left.\left. + \left(\frac{\pi}{\beta\hbar\omega_0}\right)^2 - \frac{6}{\beta\mu}\left(\frac{1}{\beta\hbar\omega_0}\right)^2 F_3(-\mathrm{e}^{-\beta\mu})\right\}\right] \tag{78}$$

which implies that a correction is obtained in this case. Let us solve the above equation in order to obtain the limiting cases for χ_θ. So, equation (78) can be written in compact form

$$\chi_\theta = \chi_p\left[1 + a\lambda + \tfrac{3}{4}\lambda^2\right] \tag{79}$$

where $\lambda = (m\omega_0\theta)^2$ and $a = 1 + \left(\frac{\mu}{\hbar\omega_0}\right)^2 + 3\left(\frac{m\omega_0\theta}{2}\right)^2 + \left(\frac{\pi}{\beta\hbar\omega_0}\right)^2 - \frac{1}{\beta\mu}\left(\frac{1}{\beta\hbar\omega_0}\right)^2 F_3(-\mathrm{e}^{-\beta\mu})$. The possible solutions of equation (79) are

$$\lambda_\pm = \tfrac{2}{3}\left(-a \pm \sqrt{a^2-3}\right) \tag{80}$$

we can see that at $\lambda_\pm$ values, the susceptibility vanishes. However for $\lambda \in]\lambda_-, \lambda_+[$, there is a diamagnetic behaviour, but otherwise the system exibits a paramagnetic behaviour. Now by switching off the noncommutative parameter, we get

$$\chi_\theta \equiv \chi_p = \mu\left(\frac{e}{2mc\omega_0}\right)^2 \tag{81}$$

this shows that in the commutative case, the system exibits an orbital paramagnetism in the limiting case for magnetic field [13].

5.3. Fermi–Dirac trace formulas

It is well known that, like the Gaussian function, the function $\operatorname{sech} x = 1/\cosh x$ is a fixed point of the Fourier transform in the Schwartz space:

$$\frac{1}{\cosh \sqrt{\frac{\pi}{2}}x} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-\mathrm{i}xy}}{\cosh \sqrt{\frac{\pi}{2}}y} \,\mathrm{d}y. \tag{82}$$

Then for a given Hamiltonian H, the Fermi operator is

$$f(H) \equiv \frac{1}{1 + \mathrm{e}^{\beta(H-\mu)}} = \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-(\mathrm{i}k+1)\frac{\beta}{2}(H-\mu)}}{4\cosh \frac{\pi}{2}k} \,\mathrm{d}k \tag{83}$$

and the corresponding thermodynamical potential operator takes the form

$$-\frac{1}{\beta}\log\left(1 + \mathrm{e}^{-\beta(H-\mu)}\right) = -\frac{1}{\beta}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-(\mathrm{i}k+1)\frac{\beta}{2}(H-\mu)}}{\left(2\cosh\frac{\pi}{2}k\right)(\mathrm{i}k+1)} \,\mathrm{d}k. \tag{84}$$

Therefore, the average number of fermions and the thermodynamical potential can be written as

$$\langle N\rangle = \mathrm{Tr} f(H) = \int_{-\infty}^{+\infty} \frac{\mathrm{e}^{(\mathrm{i}k+1)\frac{\beta\mu}{2}}}{4\cosh\frac{\pi}{2}k}\Theta(k)\,\mathrm{d}k \tag{85}$$

$$\Omega = \mathrm{Tr}\left(-\frac{1}{\beta}\log\left(1 + \mathrm{e}^{-\beta(H-\mu)}\right)\right) = -\frac{1}{\beta}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{(\mathrm{i}k+1)\frac{\beta\mu}{2}}}{\left(2\cosh\frac{\pi}{2}k\right)(\mathrm{i}k+1)}\Theta(k)\,\mathrm{d}k \tag{86}$$

where Θ designates the function

$$\Theta(k) = \mathrm{Tr}\left(\mathrm{e}^{-(\mathrm{i}k+1)\frac{\beta}{2}H}\right). \tag{87}$$

Observe that $(2n+1)\mathrm{i}$, $n \in \mathbb{Z}$, are (simple) poles for the function $1/\cosh\frac{\pi}{2}k$ and i is a pole for the functions $\Theta(k)$ and $1/(\mathrm{i}k+1)$. These Fourier integrals can be evaluated by using residue theorems if the integrand functions $\Phi_1(k) = \Theta(k)/\cosh\frac{\pi}{2}k$ and $\Phi_2(k) = \Theta(k)/((\mathrm{i}k+1)\cosh\frac{\pi}{2}k)$ satisfy the Jordan Lemma, that is, $\Phi_1(r\mathrm{e}^{\mathrm{i}\varphi}) \leqslant g(r)$, $\Phi_2(r\mathrm{e}^{\mathrm{i}\varphi}) \leqslant h(r)$, for all $\varphi \in [0,\pi]$, and $g(r)$ and $h(r)$ vanish as $r \to \infty$. The quantities $\langle N \rangle$ and Ω are then formally given by

$$2\pi\mathrm{i}\left[a_{-1}(\mathrm{i}) + \sum_{n=1}^{\infty} a_{-1}((2n+1)\mathrm{i}) + \sum_{\nu} a_{-1}(k_\nu)\right] \tag{88}$$

where $a_{-1}(\cdot)$ denotes the residue of the involved integrand at pole $(\cdot)$, and the k_ν's are the poles (with the exclusion of the pole i) of $\Theta(k)$ in the complex k-plane.

We now apply the above tools to get the thermodynamical potential through Fermi–Dirac trace formulas. To do that, we begin by evaluating equation (87) at noncommutative level. Then, in our case we can write $\Theta(k)$ as follows:

$$\tilde{\Theta}(k) = \mathrm{e}^{-(\mathrm{i}k+1)\frac{\beta}{4}\hbar\tilde{\omega}} \frac{1}{1 - \mathrm{e}^{-(\mathrm{i}k+1)\frac{\beta}{2}\hbar\tilde{\omega}_+}} \frac{1}{1 - \mathrm{e}^{-(\mathrm{i}k+1)\frac{\beta}{2}\hbar\tilde{\omega}_-}}. \tag{89}$$

Subsequently, the Fourier integral representation for the thermodynamical potential equation (86) becomes

$$\Omega_\theta = -\frac{1}{\beta}\int_{-\infty}^{+\infty} \frac{\mathrm{e}^{-(\mathrm{i}k+1)\frac{\beta}{2}\left(\frac{\hbar\tilde{\omega}}{2}-\mu\right)}}{2\cosh\frac{\pi}{2}k}\left(\frac{1}{\mathrm{i}k+1}\right)\left(\frac{1}{1 - e^{-(\mathrm{i}k+1)\frac{\beta}{2}\hbar\tilde{\omega}_+}}\right)\left(\frac{1}{1 - e^{-(\mathrm{i}k+1)\frac{\beta}{2}\hbar\tilde{\omega}_-}}\right)\mathrm{d}k. \tag{90}$$

As indicated in the formula (87), this Fourier integral is given as a series by using the residue theorem. One can easily see that the numbers $(2n+1)\mathrm{i}$, $n \in \mathbb{Z}$, are simple poles of $\operatorname{sech}\frac{\pi}{2}k$, i is a double pole of $\Theta\,(k)$, and $\mathrm{i}+4\pi n/(\beta\hbar\tilde{\omega}_+)$, $\mathrm{i}+4\pi n/(\beta\hbar\tilde{\omega}_-)$, $n\in\mathbb{Z}^*$, are simple or double poles of $\Theta(k)$ according to whether $\tilde{\omega}_+$ and $\tilde{\omega}_-$ are uncommensurable or not. In order to fulfill the requirements of the Jordan Lemma, one has to consider the following two cases: $\mu\leqslant\hbar\tilde{\omega}/2$ and $\mu\geqslant\hbar\tilde{\omega}/2$. In the first case we take an integration path lying in the lower half-plane and involving only the simple poles $(2n+1)\mathrm{i}$, $n<0$. We get

$$\Omega_\theta=\frac{1}{4\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\mathrm{e}^{\beta\mu n}}{\sinh\left(\frac{\beta}{2}\hbar\tilde{\omega}_+n\right)\sinh\left(\frac{\beta}{2}\hbar\tilde{\omega}_-n\right)}. \tag{91}$$

In the second case, an integration path in the upper half-plane is chosen. It encircles all the other poles: $(2n+1)\mathrm{i}$, $n\geqslant 0$, $\mathrm{i}+4\pi n/(\beta\hbar\tilde{\omega}_+)$, $\mathrm{i}+4\pi n/(\beta\hbar\tilde{\omega}_-)$, $n\in\mathbb{Z}^*$. We present the result in a manner which will render apparent the various regimes:

$$\begin{aligned}\Omega_\theta&=(\tilde{\Omega}_L+\tilde{\Omega}_{01})+\tilde{\Omega}_{02}+\tilde{\Omega}_{\mathrm{osc}}\\ &=2\pi\mathrm{i}\left(\overbrace{a_{-1}(\mathrm{i})}+\overbrace{\sum_{n\geqslant 1}a_{-1}((2n+1)\mathrm{i})}+\overbrace{\sum_{n_\pm\neq 0}(a_{-1}(\mathrm{i}+\frac{4\pi}{\beta\hbar\tilde{\omega}_\pm}n_\pm)}\right).\end{aligned} \tag{92}$$

Here we suppose that $m\omega_c\theta\neq -2$ is satisfied and as mentioned before the opposite case will be considered in the last section. For $\tilde{\Omega}_L$, we find

$$\tilde{\Omega}_L=\frac{\mu}{6\omega_0^2}\left(\frac{\omega_c+\left(m\omega_c^2\theta/4\right)-m\omega_0^2\theta}{2+m\omega_c\theta}\right)^2 \tag{93}$$

and $\tilde{\Omega}_{01}$ can be written as follows:

$$\tilde{\Omega}_{01}=-\frac{2\mu}{(2+m\omega_c\theta)^2}\left[\left(\frac{\mu}{\hbar\omega_0}\right)^2+\left(\frac{\pi}{\beta\hbar\omega_0}\right)^2\right]+\frac{\mu}{12}. \tag{94}$$

$\tilde{\Omega}_{02}$ is given by

$$\tilde{\Omega}_{02}=\frac{1}{4\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\frac{\exp\left(-\beta\mu n\right)}{\sinh\left(\frac{\beta\hbar\tilde{\omega}_+}{2}n\right)\sinh\left(\frac{\beta\hbar\tilde{\omega}_-}{2}n\right)}. \tag{95}$$

For $\tilde{\omega}_+/\tilde{\omega}_-\notin\mathbb{Q}$, we obtain

$$\begin{aligned}\tilde{\Omega}_{\mathrm{osc}}&=\frac{1}{2\beta}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left[\frac{\sin\left(\frac{2\mu}{\hbar\tilde{\omega}_-}\pi n\right)}{\sin\left(\frac{\tilde{\omega}_+}{\omega_-}\pi n\right)\sinh\left(\frac{2\pi^2 n}{\beta\hbar\tilde{\omega}_-}\right)}+\frac{\sin\left(\frac{2\mu}{\hbar\tilde{\omega}_+}\pi n\right)}{\sin\left(\frac{\tilde{\omega}_-}{\tilde{\omega}_+}\pi n\right)\sinh\left(\frac{2\pi^2 n}{\hbar\tilde{\omega}_+}\right)}\right]\\ &\equiv\tilde{\Omega}_{\mathrm{osc}}^-+\tilde{\Omega}_{\mathrm{osc}}^+.\end{aligned} \tag{96}$$

However, for $\tilde{\omega}_+/\tilde{\omega}_-=p/q\in\mathbb{Q}$, $\gcd(p,q)=1$, $\tilde{\omega}_+/p=\tilde{\omega}_-/q=2l/(\hbar\beta)\in\mathbb{R}$, we have

$$\begin{aligned}\tilde{\Omega}_{\mathrm{osc}}=\frac{1}{2\beta}\Bigg[&\sum_{n=1,n\not\equiv 0\bmod q}^{\infty}\frac{(-1)^n}{n}\frac{\sin\left(\frac{2\mu}{\hbar\tilde{\omega}_-}\pi n\right)}{\sin\left(\frac{\tilde{\omega}_+}{\omega_-}\pi n\right)\sinh\left(\frac{2\pi^2 n}{\beta\hbar\tilde{\omega}_-}\right)}\\ &+\sum_{n=1,m\not\equiv 0\bmod p}^{\infty}\frac{(-1)^n}{n}\frac{\sin\left(\frac{2\mu}{\hbar\tilde{\omega}_+}\pi n\right)}{\sin\left(\frac{\tilde{\omega}_-}{\tilde{\omega}_+}\pi n\right)\sinh\left(\frac{2\pi^2 n}{\hbar\tilde{\omega}_+}\pi^2 n\right)}\\ &+\frac{1}{lpq}\sum_{k=1}^{\infty}\frac{(-1)^{(p+q)k}}{k\sinh\left(\frac{\pi^2}{l}k\right)}\left[\beta\mu\cos\left(\frac{\beta\mu\pi k}{l}\right)\right.\\ &\left.-\left(\pi\coth\left(\frac{\pi^2}{l}k\right)+\frac{l}{\pi k}\right)\sin\left(\frac{\beta\mu\pi k}{l}\right)\right]\Bigg].\end{aligned} \tag{97}$$

Therefore, the average number of electrons is

$$
\begin{aligned}
\langle \tilde{N}_e \rangle = & -\frac{1}{6\omega_0^2}\left(\frac{\omega_c + \left(m\omega_c^2\theta/4\right) - m\omega_0^2\theta}{2+m\omega_c\theta}\right)^2 \\
& + \frac{1}{2}\left[\frac{4}{(2+m\omega_c\theta)^2}\left(\left(\frac{\mu}{\hbar\omega_0}\right)^2 + \frac{1}{3}\left(\frac{\pi}{\hbar\omega_0}\right)^2\right) - \frac{1}{6}\right] \\
& + \frac{1}{4}\sum_{n=1}^{\infty}(-1)^n \frac{\mathrm{e}^{-\beta\mu n}}{\sinh\left(\frac{\beta}{2}\hbar\tilde{\omega}_+ n\right)\sinh\left(\frac{\beta}{2}\hbar\tilde{\omega}_- n\right)} - \pi\sum_{n=1}^{\infty}(-1)^n \\
& \times\left[\frac{1}{\beta\hbar\tilde{\omega}_-}\frac{\cos\left(\frac{2\mu}{\hbar\tilde{\omega}_-}\pi n\right)}{\sin\left(\frac{\tilde{\omega}_+}{\tilde{\omega}_-}\pi n\right)\sinh\left(\frac{2\pi^2 n}{\beta\hbar\tilde{\omega}_-}\right)} + \frac{1}{\beta\hbar\tilde{\omega}_+}\frac{\cos\left(\frac{2\mu}{\hbar\tilde{\omega}_+}\pi n\right)}{\sin\left(\frac{\tilde{\omega}_-}{\tilde{\omega}_+}\pi n\right)\sinh\left(\frac{2\pi^2 n}{\hbar\tilde{\omega}_+}\right)}\right] \\
\equiv & \langle \tilde{N}_e\rangle_L + \langle \tilde{N}_e\rangle_{01} + \langle \tilde{N}_e\rangle_{02} + \langle \tilde{N}_e\rangle_{\mathrm{osc}}^- + \langle \tilde{N}_e \rangle_{\mathrm{osc}}^+
\end{aligned} \tag{98}
$$

and the magnetic moment can be written as follows:

$$
M_\theta = \tilde{M}_L + \tilde{M}_{01} + \tilde{M}_{02} + \tilde{M}_{\mathrm{osc}}^- + \tilde{M}_{\mathrm{osc}}^+ \tag{99}
$$

where

$$
\begin{aligned}
\tilde{M}_L &= -\frac{e\mu}{3mc\omega_0^2}\left(\frac{\omega_c + \left(m\omega_c^2\theta/4\right) - m\omega_0^2\theta}{2+m\omega_c\theta}\right)\left[\frac{1}{2} - \frac{\left(\omega_c + \left(m\omega_c^2\theta/4\right) - m\omega_0^2\theta\right)m\theta}{(2+m\omega_c\theta)^2}\right] \\
\tilde{M}_{01} &= -\frac{4e\mu}{3mc}\frac{m\theta}{(2+m\omega_c\theta)^3}\left[\left(\frac{\mu}{\hbar\omega_0}\right)^2 + \left(\frac{\pi}{\beta\hbar\omega_0}\right)^2\right]
\end{aligned} \tag{100}
$$

and for $\tilde{M}_{02}$, we have

$$
\begin{aligned}
\tilde{M}_{02} = & \frac{\hbar e}{4mc}\sum_{n=1}^{\infty}(-1)^n\frac{\mathrm{e}^{-\beta\mu n}}{\sinh\left(n\beta\hbar\tilde{\omega}_+\right)\sinh\left(n\beta\hbar\tilde{\omega}_-\right)} \\
& \times\left[\frac{1}{\tilde{\omega}}\left(\omega_c + \frac{m\theta}{4}(2\omega_c^2+\omega^2) + 2\omega_c(m\omega\theta/4)^2\right)\right. \\
& \times\left(\coth\left(n\beta\hbar\tilde{\omega}_+\right) + \coth\left(n\beta\hbar\tilde{\omega}_-\right)\right) + \frac{1}{2}(2+m\omega_c\theta) \\
& \left.\times\left(\coth\left(n\beta\hbar\tilde{\omega}_+\right) - \coth\left(n\beta\hbar\tilde{\omega}_-\right)\right)\right]
\end{aligned} \tag{101}
$$

and, for the irrational case $\omega_+/\omega_- \notin \mathbb{Q}$,

$$
\begin{aligned}
\tilde{M}_{\mathrm{osc}}^- = & \frac{e\pi}{\beta mc}\sum_{n=1}^{\infty}\frac{(-1)^n}{\sin\left(\pi n\frac{\tilde{\omega}_+}{\tilde{\omega}_-}\right)\sinh\left(\frac{2\pi^2 n}{\beta\hbar\tilde{\omega}_-}\right)}\left[\frac{1}{\tilde{\omega}}\left(\omega_c + \frac{m\theta}{4}\left(2\omega_c^2+\omega^2\right) + 2\omega_c(m\omega\theta/4)^2\right)\right. \\
& \times\left(\frac{\mu}{\hbar\tilde{\omega}_-^2}\cos\left(2\pi n\frac{\mu}{\hbar\tilde{\omega}_-}\right) - \frac{\tilde{\omega}_c}{\tilde{\omega}_-^2}\cot\left(\pi n\frac{\tilde{\omega}_+}{\tilde{\omega}_-}\right)\sin\left(2\pi n\frac{\mu}{\hbar\tilde{\omega}_-}\right)\right. \\
& \left.- \frac{\pi}{\beta\hbar\tilde{\omega}_-^2}\sin\left(\pi n\frac{\tilde{\omega}_+}{\tilde{\omega}_-}\right)\coth\left(\frac{2\pi^2 n}{\beta\hbar\tilde{\omega}_-}\right)\right) + \frac{1}{2}(2+m\omega_c\theta) \\
& \times\left(-\frac{\mu}{\hbar\tilde{\omega}_-^2}\cos\left(2\pi n\frac{\mu}{\hbar\tilde{\omega}_-}\right) + \frac{\tilde{\omega}}{\tilde{\omega}_-^2}\cot\left(\pi n\frac{\tilde{\omega}_+}{\tilde{\omega}_-}\right)\sin\left(2\pi n\frac{\mu}{\hbar\tilde{\omega}_-}\right)\right. \\
& \left.\left.+ \frac{\pi}{\beta\hbar\tilde{\omega}_-^2}\sin\left(\pi n\frac{\tilde{\omega}_+}{\tilde{\omega}_-}\right)\coth\left(\frac{2\pi^2 n}{\beta\hbar\tilde{\omega}_-}\right)\right)\right]
\end{aligned} \tag{102}
$$

and the same result can be obtained for $\tilde{M}^{+}_{\rm osc}$

$$\tilde{M}^{+}_{\rm osc} = \frac{e\pi}{\beta mc}\sum_{n=1}^{\infty}\frac{(-1)^n}{\sin\left(\pi n\frac{\tilde{\omega}_+}{\tilde{\omega}_-}\right)\sinh\left(\frac{2\pi^2 n}{\beta\hbar\tilde{\omega}_-}\right)}\left[\frac{1}{\tilde{\omega}}\left(\omega_c + \frac{m\theta}{4}\left(2\omega_c^2+\omega^2\right)\right.\right.$$
$$+2\omega_c(m\omega\theta/4)^2\Big)\left(\frac{\mu}{\hbar\tilde{\omega}_-^2}\cos\left(2\pi n\frac{\mu}{\hbar\tilde{\omega}_+}\right) - \frac{\tilde{\omega}_c}{\tilde{\omega}_+^2}\cot\left(\pi n\frac{\tilde{\omega}_-}{\tilde{\omega}_+}\right)\right.$$
$$\times\sin\left(2\pi n\frac{\mu}{\hbar\tilde{\omega}_+}\right) - \frac{\pi}{\beta\hbar\tilde{\omega}_+^2}\sin\left(\pi n\frac{\tilde{\omega}_-}{\tilde{\omega}_+}\right)\coth\left(\frac{2\pi^2 n}{\beta\hbar\tilde{\omega}_+}\right)\Big) + \frac{1}{2}(2+m\omega_c\theta)$$
$$\times\left(-\frac{\mu}{\hbar\tilde{\omega}_+^2}\cos\left(2\pi n\frac{\mu}{\hbar\tilde{\omega}_+}\right) + \frac{\tilde{\omega}}{\tilde{\omega}_+^2}\cot\left(\pi n\frac{\tilde{\omega}_-}{\tilde{\omega}_+}\right)\sin\left(2\pi n\frac{\mu}{\hbar\tilde{\omega}_+}\right)\right.$$
$$\left.\left.+\frac{\pi}{\beta\hbar\tilde{\omega}_+^2}\sin\left(\pi n\frac{\tilde{\omega}_-}{\tilde{\omega}_+}\right)\coth\left(\frac{2\pi^2 n}{\beta\hbar\tilde{\omega}_+}\right)\right)\right]. \tag{103}$$

Similar formulas can be derived for $\mathcal{M}^{\pm}_{\rm osc}$ in the rational case. These expression can be studied in different limits of temperature, magnetic field and noncommutative parameter in order to understand the behaviour of the system under consideration. This will be the subject of the forthcoming paper [14].

5.4. Critical point $m\omega_c\theta = -2$

Let us mention that this critical point is actually equivalent to $\frac{eB\theta}{c} = -2$. By using the transformation (11) and taking ($c = 1$, $e = 1$, $m = 1$), we find the critical point $B\theta = 1$ obtained in [19].

By taking $m\omega_c\theta = -2$, the set of frequencies defined in section 3.1 becomes

$$\tilde{\omega} = -\frac{\omega^2}{2\omega_c} \qquad \tilde{\omega}_c = \frac{\omega_c}{2}\left(1+\frac{4\omega_0^2}{\omega_c^2}\right)$$
$$\tilde{\omega}_+ = 0 \qquad \tilde{\omega}_- = 2\tilde{\omega}. \tag{104}$$

Now if we come back to equation (21), we get a Hamiltonian of a harmonic oscillator of frequency $\tilde{\omega}$, such that

$$H_\theta\left(\theta = -\frac{2}{m\omega_c}\right) = \frac{\hbar\tilde{\omega}}{2}(2\tilde{N}_g + 1) \tag{105}$$

where the eigenstates and the eigenvalues are $|\tilde{n}_g\rangle$ and $\frac{\hbar\tilde{\omega}}{2}(2\tilde{n}_g + 1)$, respectively. Therefore, the thermodynamical potential equation (44) can now be written in terms of $H_\theta\left(\theta = -\frac{2}{m\omega_c}\right)$

$$\Omega_\theta\left(\theta = -\frac{2}{m\omega_c}\right) = -\frac{1}{\beta}Tr\, e^{-\beta(H_\theta(\theta=-2/m\omega_c)-\mu)} \tag{106}$$

where the corresponding partition function is

$$Z_\theta\left(\theta = -\frac{2}{m\omega_c}\right) = \mathrm{Tr}\, e^{-\beta(H_\theta(\theta=-2/m\omega_c)-\mu)} \tag{107}$$

and the trace is taken on the eigenstates $|\tilde{n}_g\rangle$. Actually, we can construct coherent states in such a way that

$$|\tilde{z}_g\rangle = \exp\left[-\frac{1}{2}|\tilde{z}_g|^2\right] e^{\tilde{z}_g\tilde{a}_g^\dagger}|\tilde{0}\rangle$$
$$\tilde{a}_g|\tilde{z}_g\rangle = \tilde{z}_g|\tilde{z}_g\rangle. \tag{108}$$

With respect to the last equation, $Z_\theta\left(\theta=-\frac{2}{m\omega_c}\right)$ can be expressed as follows:

$$Z_\theta\left(\theta=-\frac{2}{m\omega_c}\right)=\mathrm{e}^{\beta\mu}\int \mathrm{d}^2\tilde{z}_g\langle\tilde{z}_g|\mathrm{e}^{-\beta\frac{\hbar\tilde{\omega}}{2}(2\tilde{N}_g+1)}|\tilde{z}_g\rangle. \tag{109}$$

To calculate the partition function, one can consider the boson-operator identity [25]

$$\mathrm{e}^{\xi a^\dagger a}=\sum_{n=0}^{\infty}\frac{(\mathrm{e}^{\xi}-1)^n}{n!}a^\dagger a \tag{110}$$

which holds for any operators $a^\dagger$ and a satisfying the commutation relation $[a,a^\dagger]=1$. By applying this identity, we can show that

$$Z_\theta\left(\theta=-\frac{2}{m\omega_c}\right)=\mathrm{e}^{-\beta(\frac{\hbar\tilde{\omega}}{2}-\mu)}\int \mathrm{d}^2\tilde{z}_g\,\mathrm{e}^{-|\tilde{z}_g|^2(1-\mathrm{e}^{-\beta\hbar\tilde{\omega}})}. \tag{111}$$

After integration, we obtain

$$Z_\theta\left(\theta=-\frac{2}{m\omega_c}\right)=\frac{\mathrm{e}^{\beta\mu}}{4\sinh\left(\frac{\beta\hbar\tilde{\omega}}{2}\right)}. \tag{112}$$

Thus, the thermodynamical potential becomes

$$\Omega_\theta\left(\theta=-\frac{2}{m\omega_c}\right)=-\frac{1}{\beta}\log\left(4\sinh\left(\frac{\beta\hbar\tilde{\omega}}{2}\right)\right)-\mu. \tag{113}$$

We get for the magnetic moment

$$M_\theta\left(\theta=-\frac{2}{m\omega_c}\right)=-\frac{e\hbar}{4mc}\frac{\omega^2}{\omega_c^2}\coth\left(\frac{\beta\hbar\omega^2}{4\omega_c}\right) \tag{114}$$

and hence the susceptibility is

$$\chi_\theta\left(\theta=-\frac{2}{m\omega_c}\right)=\frac{1}{2}\left(\frac{e\hbar}{2mc}\right)^2\left[\frac{1}{\hbar\omega_c}\left(\frac{\omega^2}{\omega_c^2}-1\right)\coth\left(\frac{\beta\hbar\omega^2}{4\omega_c}\right)\right.$$
$$\left.+\frac{\beta}{2}\frac{\omega^4}{\omega_c^4}\left(1+\coth^2\left(\frac{\beta\hbar\omega^2}{4\omega_c}\right)\right)\right]. \tag{115}$$

From the last equation, we observe that susceptibility becomes infinite at zero magnetic field. Note that there are some physical systems where infinite susceptibility is actually seen [26].

6. Conclusion

We have investigated the Fock–Darwin Hamiltonian on the noncommutative space. We started by giving a noncommutative version of this Hamiltonian. Subsequently, the eigenstates and the corresponding eigenvalues has been derived through two methods, an algebraic and an analytical. The degeneracy of Landau levels has been considered and some algebras: $su(2)$ and $su(1,1)$ have been realized. In particular it has been shown that the degeneracy of Landau levels can be lifted for this model at weak magnetic field limit. Using the Berezin–Lieb inqualities, we have obtained the magnetic behaviour of this model at high temperature, which is absent in the commutative case. For low temperature, a θ-dependent correction to susceptibility has been pointed out. Furthermore, through the use of the Fermi–Dirac trace formulas, a generalization of the thermodynamical potential, the average number of electrons and the magnetic moment have been found in terms of the noncommutative parameter. At critical point, by using another approach, the magnetic moment and susceptibility have been obtained.

Finally, we mention that this generalization can be studied in various regimes of temperature, magnetic field and noncommutative parameter. We could also think to investigate the relationship between the spatial distribution of current and the magnetic moment of the whole system at the noncommutative level. Another possibility is to study the results derived in this paper numerically. We hope to return to these questions in a subsequent publication.

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